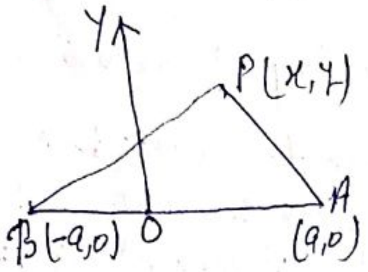
B.Sc-part-I (H), Paper-II Date-14-04-2020

Q.9: Prove that the locus of a point which moves such that the sum of the distances from two fixed points is constant, is an ellipse.

Soln:




Let the fixed points be A and B.

Let O be the middle point of AB.

Let OY be $\perp$ to OA. Let OA be the x-axis, OY the y-axis and O the origin. Let $AB = 2a$

Then A is the point $(a,0)$ and B is the point $(-a,0)$. Let P be the variable point $(x,y)$.

By question, $PA + PB = 2k$ (say) $\xrightarrow{}$ I, where $k$ is a constant.

But $PA^2 = (x-a)^2 + y^2$ — II and $PB^2 = (x+a)^2 + y^2$ — (III)

We note that from the triangle PAB, the sum of the sides PA and PB is greater than the third side AB

i.e. $2a < 2k$

By I, II and III

$$\sqrt{(x-a)^2+y^2} + \sqrt{(x+a)^2+y^2} = 2k$$

$$\Rightarrow \sqrt{(x-a)^2+y^2} = 2k - \sqrt{(x+a)^2+y^2}$$

Squaring both sides, we get

$$(x-a)^2+y^2 = 4k^2 + (x+a)^2 + y^2 - 4k\sqrt{(x+a)^2+y^2}$$

$$\Rightarrow 4k^2 + 4ax = 4k\sqrt{(x+a)^2+y^2}$$

Again, squaring both sides,

$$(k^2+ax)^2 = k^2[(x+a)^2+y^2]$$

$$\Rightarrow k^4 + a^2x^2 + 2k^2ax = k^2(x+a)^2 + k^2y^2$$

$$\Rightarrow x^2(k^2-a^2) + k^2y^2 = k^2(k^2-a^2)$$

Dividing through by $k^2(k^2-a^2)$,

$$\frac{x^2}{k^2} + \frac{y^2}{k^2-a^2} = 1$$

This is the eqn of an ellipse whose semi-major axis is $k$ and semiminor axis is $\sqrt{k^2-a^2}$.

Lecturered By Dr. H.K. Jadav.
Deptt. of Mathematics
Marwari College, Darbhanga.
~~[illegible]~~ Date – 14-04-2020.
B.Sc.-part-I(H), paper-II

Analytical Geometry of two dimensions.

Q.1. Through what angle must the axes be turned, without change of origin, so that the expression $ax^2+2hxy+by^2$ may become an expression in which there is no term involving $x'y'$, where $(x', y')$ is any point referred to the new axes?

Soln. Let the axes be turned through an angle $\theta$ then
$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

Therefore, the given expression $ax^2+2hxy+by^2$ changes into

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + b(x'\sin\theta + y'\cos\theta)$$

$$\Rightarrow x'^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) + 2x'y'(-a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta) + y'^2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta).$$

By question, this expression should not contain the term $x'y'$.

$$\therefore -a\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) + b\sin\theta\cos\theta = 0$$

$$\Rightarrow 2h\cos 2\theta = (a-b)\sin 2\theta$$

$$\Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = \frac{2h}{a-b}$$

$$\Rightarrow \tan 2\theta = \frac{2h}{a-b}$$

Hence, $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$.

Lecturered by Dr. H.R. Yadav
Deptt. of Mathematics
Marwari College, Darbhanga.
B.Sc.-Part-II (H), Paper-III
Date - 14-04-2020.

# Differential Calculus-

Q.1. State and prove Maclaurin's theorem to expand $f(x)$.

Soln. Statement:→ Let the function $f(x)$ can be expanded in powers of $x$, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \dots \text{ to } \infty.$$

Proof: Let us consider the possibility of expanding $f(x)$ in a convergent series of positive integral powers of $x$, so that

$$f(x) = A_0 + A_1 x + A_2 \frac{x^2}{\lfloor 2} + A_3 \frac{x^3}{\lfloor 3} + A_4 \frac{x^4}{\lfloor 4} + \dots \text{ to } \infty \quad \text{--- (I)}$$

Where, $A_0, A_1, A_2, A_3, \dots$ are constants to be determined, not containing $x$.

Differentiating (I) with respect to $x$, we get

$$f'(x) = A_1 + A_2 \frac{2x}{\lfloor 2} + A_3 \frac{3x^2}{\lfloor 3} + A_4 \frac{4x^3}{\lfloor 4} + \dots \text{ to } \infty$$

$$\Rightarrow f'(x) = A_1 + A_2 x + A_3 \frac{x^2}{\lfloor 2} + A_4 \frac{x^3}{\lfloor 3} + \dots \text{ to } \infty \quad \text{--- II}$$

Differentiating II with respect to $x$, we get

$$f''(x) = A_2 + A_3 \frac{2x}{\lfloor 2} + A_4 \frac{3x^2}{\lfloor 3} + \dots \text{ to } \infty.$$

$$\Rightarrow f''(x) = A_2 + A_3 x + A_4 \frac{x^2}{\lfloor 2} + \dots \text{ to } \infty \quad \text{--- III}$$

Differentiating III with respect to $x$, we get

$$f'''(x) = A_3 + A_4 \frac{2x}{\lfloor 2} + \dots \text{ to } \infty$$

$$\Rightarrow f'''(x) = A_3 + A_4 x + \dots \text{ to } \infty \quad \text{--- (IV)}$$

Putting $x = 0$ in I, II, III, IV, ... we get

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = A_2,\ f'''(0) = A_3, \dots$$

Substituting the values of these constants in I, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \dots \text{ to } \infty.$$

Proved.

**Q.2.** Expand $\cos x$ by Maclaurin's theorem.

**Soln** Let $f(x)=\cos x$

Differentiating w.r. to $x$, $f'(x)=-\sin x$

$f''(x)=-\cos x$, $f'''(x)=\sin x$, $f^{IV}(x)=\cos x$, $f^{V}(x)=-\sin x$

$f^{VI}(x)=-\cos x$ and so on.

At $x=0$, $f(0)=1$, $f'(0)=0$, $f''(0)=-1$, $f'''(0)=0$, $f^{IV}(0)=1$, $f^{V}(0)=0$, $f^{VI}(0)=-1$ and so on.

By Maclaurin's theorem,

$$f(x)=f(0)+x f'(0)+\frac{x^2}{\lfloor 2}f''(0)+\frac{x^3}{\lfloor 3}f'''(0)+\frac{x^4}{\lfloor 4}f^{IV}(0)+\frac{x^5}{\lfloor 5}f^{V}(0)+\frac{x^6}{\lfloor 6}f^{VI}(0)+\cdots \text{ to }\infty$$

Hence, $\cos x = 1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\cdots$ to $\infty$.

**Q.3.** Apply Maclaurin's series to prove $\sin x = x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\cdots$ to $\infty$

**Soln** Let $f(x)=\sin x$

Diff. successively w.r. to $x$, $f'(x)=\cos x$

$f''(x)=-\sin x$, $f'''(x)=-\cos x$, $f^{IV}(x)=\sin x$, $f^{V}(x)=\cos x$ and so on.

At $x=0$, $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-1$, $f^{IV}(0)=0$, $f^{V}(0)=1$, ....

By Maclaurin's series,

$$f(x)=f(0)+x f'(0)+\frac{x^2}{\lfloor 2}f''(0)+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}f^{IV}(0)+\frac{x^5}{\lfloor 5}f^{V}(0)+\cdots \text{ to }\infty$$

Hence, $\sin x = x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\cdots$ to $\infty$.